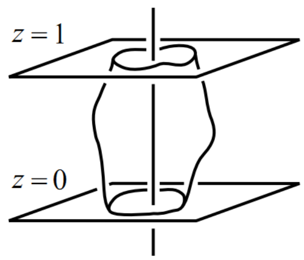
Lecture 11 - Advanced Rotational Dynamics

A Puzzle...

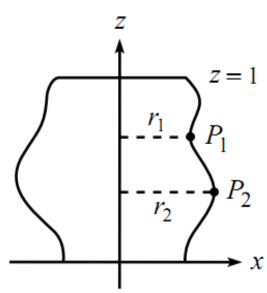
A moldable blob of matter of mass M and uniform density is to be situated between the planes $z = 0$ and $z = 1$ so that the moment of inertia around the z-axis is as small as possible. What shape should the blob take?



Solution

The shape should be a cylinder with the z-axis as its symmetry axis. How can we prove this?

Suppose for the sake of a contradiction that the optimal blob is not a cylinder. Then there must exist two points on the surface P_1 and P_2 such that their radial distances from the z-axis are r_1 and r_2 with $r_1 < r_2$.



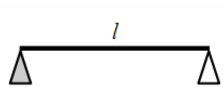
If we move a small piece of the blob from P_2 to P_1, we will lower the moment of inertia, a contradiction. Therefore, all points on the surface must be equidistant from the z-axis, and hence the blob is a cylinder. □

So You Think You Know Angular Momentum...

Removing a Support

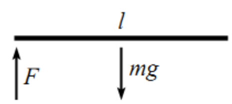
Example

A uniform rod of length l and mass m rests on supports at its ends. The right support is quickly removed. What is the force on the left support immediately thereafter?



<u>Solution</u>

Denote the force from the left support immediately after the right support has been removed as F.



Let the acceleration downwards of the stick's center of mass be a and its (clockwise) angular acceleration about its center of mass be α. Then the sum of forces, the sum of torques, and the relation between a and α that keeps the left support stationary are

$$m\,g - F = m\,a \tag{1}$$
$$F\,\frac{l}{2} = \left(\frac{1}{12}\,m\,l^2\right)\alpha \tag{2}$$
$$\frac{l}{2}\,\alpha = a \tag{3}$$

This system is straightforward to solve to obtain

$$a = \frac{3}{4}\,g \tag{4}$$
$$\alpha = \frac{3}{2}\,\frac{g}{l} \tag{5}$$
$$F = \frac{1}{4}\,m\,g \tag{6}$$

Note that the right end of the stick accelerates at $a + \alpha\,\frac{l}{2} = \frac{3}{2}\,g$ which is greater than g. This makes sense because as the stick falls downwards, the force F makes the right end rotate, accelerating it even faster. (As a fun note, this implies that if you connect a swing using a solid rod instead of the typical chain, and you swing up to the horizontal position, you will accelerate down faster than g.)

An alternative way to solve this problem is to note that with the left end fixed (for the first moment after the right support is removed), the motion can also be described as an angular acceleration α (the exact same α as above!) about the left end of the stick. Therefore, the torque equation equals

$$m\,g\,\frac{l}{2} = \left(\frac{1}{3}\,m\,l^2\right)\alpha \tag{7}$$

where we have now used the moment of inertia about the end of a rod. We can now divide the torque relation $F\,\frac{l}{2} = \left(\frac{1}{12}\,m\,l^2\right)\alpha$ found above with this relation to obtain $F = \frac{1}{4}\,m\,g$. □

Lengthening the String

Pure Rolling Motion

One important mode of motion we have already encountered several times is *<u>pure rolling motion</u>*, also known as *<u>rolling without slipping</u>*. One important fact about pure rolling motion:

When rolling without slipping, the velocity of every point touching the ground is zero. (12)

At first, this may sound puzzling, as you may imagine $v = 0$ forcing the object to stop rolling, but the acceleration of these contact points need not be zero, and that is what ultimately saves the day. This is the same reason why it is *static friction* rather than kinetic friction that matters in pure rolling motion (for example, when your car drives on a road).

To be quantitative, consider a cylinder of radius R with uniform mass density rolling without slipping. As with any extended object, the motion of the rolling cylinder is fully determined if we specify the velocity V_{CM} of its center of mass and the angular velocity ω_{CM} about its center of mass. First, we analyze both of these motions separately.

Note that if either $\omega_{\text{CM}} > \frac{v_{\text{CM}}}{R}$ or $\omega_{\text{CM}} < \frac{v_{\text{CM}}}{R}$, the disk will slip while it moves, since they velocity of the base point will be nonzero. Visually, the emotion of the disk will look a bit unnatural in either case.

Now let us consider the velocity of a general point (r, θ) from the center of the cylinder. The pure translational motion will translate this point to the right with velocity $v_{\text{CM}}\,\hat{x}$. The pure rotational motion will impart a velocity $\omega_{\text{CM}}\, r\left(-\hat{\theta}\right) = \omega_{\text{CM}}\, r\,(\text{Sin}[\theta]\,\hat{x} - \text{Cos}[\theta]\,\hat{y})$ in the clockwise direction. Therefore, the net velocity of a particle is given by

$$\vec{v} = (v_{\text{CM}} + \omega_{\text{CM}}\, r\,\text{Sin}[\theta])\,\hat{x} - (\omega_{\text{CM}}\, r\,\text{Cos}[\theta])\,\hat{y} \tag{13}$$

The criteria for rolling without slipping is that the point in contact with the ground ($r = R$, $\theta = \frac{3\pi}{2}$) has zero velocity, which requires

$$v_{\text{CM}} = \omega_{\text{CM}}\, R \tag{14}$$

With this, the velocity formula simplifies to

$$\begin{aligned}\vec{v} &= (\omega_{\text{CM}}\, R + \omega_{\text{CM}}\, r\,\text{Sin}[\theta])\,\hat{x} - (\omega_{\text{CM}}\, r\,\text{Cos}[\theta])\,\hat{y} \\ &= \omega_{\text{CM}}\,\{(R + r\,\text{Sin}[\theta])\,\hat{x} - (r\,\text{Cos}[\theta])\,\hat{y}\}\end{aligned} \tag{15}$$

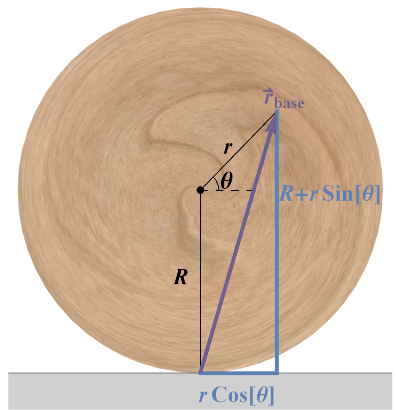
Interestingly, the position vector from the base point to (r, θ) is given by

$$\vec{r}_{\text{base}} = r\,\text{Cos}[\theta]\,\hat{x} + (R + r\,\text{Sin}[\theta])\,\hat{y} \tag{16}$$

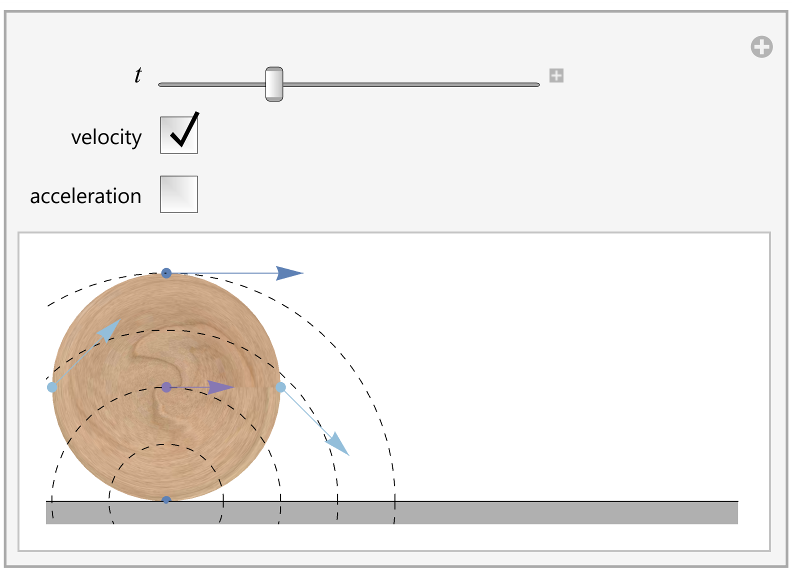
Note that the velocity and the position from the base point satisfy two important properties:

$$\vec{v} \cdot \vec{r}_{\text{base}} = 0 \tag{17}$$

$$|\vec{v}| = \omega_{\text{CM}}\,|\vec{r}_{\text{base}}| \tag{18}$$



The first implies that the two vectors are perpendicular, while the second states that the velocity of the point (r, θ) scales linearly with its distance from the base point. This implies that the velocities of each point on the rolling cylinder can be determined by considering an angular velocity vector $\vec{\omega}$ going into the page along the base point of the cylinder (and this axis moves to the right along with the cylinder).

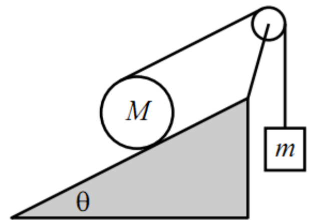


Rolling Down a Plane

<u>Example</u>

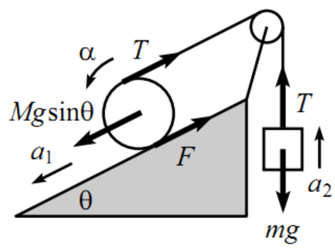
A string wraps around a uniform cylinder of mass M and radius R, which rests on a fixed plane at angle θ. The string passes up over a massless pulley and is connected to a mass m. Assume that the cylinder rolls without slipping on the plane, and that the string is parallel to the plane.

1. What is the acceleration of the mass m?
2. What must the ratio of masses $\frac{M}{m}$ be for the cylinder to accelerate down the plane?



Solution

The relevant forces are friction, tension, and gravity. (There is also a normal force, but it only serves to keep the cylinder on the plane, so we will ignore it.) Let us define the accelerations a_1 and a_2 of cylinder M and mass m, respectively, as well as the angular acceleration α_1 of cylinder M as in this diagram.



The moment of inertia of a cylinder about its center of mass equals $I = \frac{1}{2} M R^2$. Our unknowns are a_1, a_2, α, T, and F, and we can immediately write the force equations on our two masses (ignoring the irrelevant normal force on cylinder M)

$$T - m g = m a_2 \tag{19}$$

$$M g \operatorname{Sin}[\theta] - T - F = M a_1 \tag{20}$$

We can calculate the torque on the cylinder M about its center of mass

$$(F - T) R = \left(\frac{1}{2} M R^2\right) \alpha \tag{21}$$

The rolling without slipping condition implies that

$$a_1 = \alpha R \tag{22}$$

To solve for our 5 unknowns, we need one more equation in addition to the 4 above. The fifth equation comes from conserving the string, and it is (as usual) the trickiest equation. One way to think about it is that the top of a rolling wheel moves twice as fast as the center, and by differentiating with respect to time we find

$$a_2 = 2 a_1 \tag{23}$$

Another way to derive this result is to imagine the two masses starting from rest. In a time Δt, the small mass will rise up by $\frac{1}{2} a_2 (\Delta t)^2$, giving that much more string to the cylinder. The cylinder will both translate down the hill (by an amount $\frac{1}{2} a_1 (\Delta t)^2$) and rotate around its center (by an amount $\frac{1}{2} R \alpha (\Delta t)^2$). Adding these two contributions shows that it has pulled $\frac{1}{2} a_1 (\Delta t)^2 + \frac{1}{2} R \alpha (\Delta t)^2 = \frac{1}{2} (2 a_1) (\Delta t)^2$ amount of string. Because the amount of string is conserved, $a_2 = 2 a_1$.

Now that we have our system of 5 equations, it is straightforward to solve it. We expedite the matter with *Mathematica*

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Simplify@Solve[{T - m g == m a2, M g Sin[θ] - T - F == M a1, (F - T) R == (1/2 M R^2) α, a1 == α R, a2 == 2 a1}, {F, T, a1, a2, α}]
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$$\left\{\left\{F \to \frac{g M (m + (4 m + M) \mathrm{Sin}[\theta])}{8 m + 3 M},\ T \to \frac{g m M (3 + 4 \mathrm{Sin}[\theta])}{8 m + 3 M},\ a1 \to \frac{-4 g m + 2 g M \mathrm{Sin}[\theta]}{8 m + 3 M},\ a2 \to \frac{4 (-2 g m + g M \mathrm{Sin}[\theta])}{8 m + 3 M},\ \alpha \to \frac{-4 g m + 2 g M \mathrm{Sin}[\theta]}{8 m R + 3 M R}\right\}\right\}$$

Therefore, the acceleration of the small mass equals

$$a_2 = \frac{4 (M g \mathrm{Sin}[\theta] - 2 m g)}{3 M + 8 m} \tag{24}$$

In order for the cylinder to accelerate down the plane, we must have $a_2 > 0$ which implies

$$M \mathrm{Sin}[\theta] > 2 m \tag{25}$$

or equivalently a ratio of the masses

$$\frac{M}{m} > \frac{2}{\mathrm{Sin}[\theta]} \tag{26}$$

Simple results demand simple explanations! Is there a simple intuitive explanation for this simple result? There is, and to see it we should first ask under what conditions will the system achieve equilibrium ($a_1 = a_2 = 0$ and $\alpha = 0$)? The net forces on m must be zero, so Equation (19) tells us that $T = m g$. The net torque on M must also be zero, so Equation (21) yields $F = T = m g$. Finally, the sum of the forces on M must be zero, so by Equation (20) we find $M g \mathrm{Sin}[\theta] = T + F = 2 m g$ or $\frac{M}{m} = \frac{2}{\mathrm{Sin}[\theta]}$. Thus, for any θ, given this ratio of the masses will yield static equilibrium. Any other ratio of the masses must necessarily not yield static equilibrium, and in particular if the cylinder is made any heavier it will roll down the incline, recovering Equation (26).

Lets look at some special cases:

Case 1: $\theta \approx 0$

For small angles, the cylinder will roll down the plane if $\frac{M}{m} \to \infty$ which makes sense since the gravitational force on M becomes tiny.

Case 2: $\theta \approx \frac{\pi}{2}$

As noted above, if $M = 2 m$ then the system will be stationary with $T = F = m g$ pulling straight up on M while T pulls up on m. For $M > 2 m$, the cylinder will start to roll down the incline. □

Leaving the Sphere

Sliding to Rolling

Race to the Finish!

Mathematica Initialization